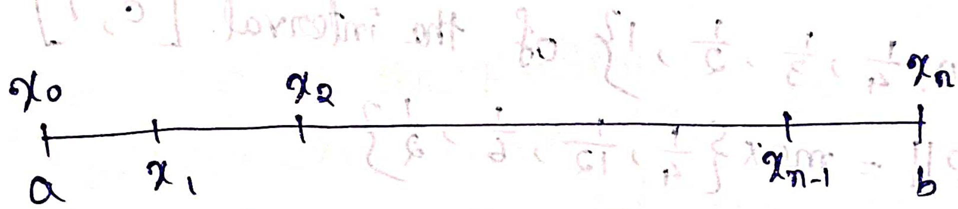
Riemann Integral

(1) **Partition :-** Let, $[a, b]$ be a closed and bounded interval. The partition of the interval $[a, b]$ is denoted by P and defined to be the set $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$, where $a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$.

Partition is also written as
$$P = \{a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b\}$$



For example,
$$P = \{0, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{1}{2}, 1\}$$ is a partition of the closed interval $[0, 1]$.

[Note :- i) The family of partitions of $[a, b]$ is denoted by $P[a, b]$.]

ii) Let, $P = \{a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b\}$ be a partition of the closed interval $[a, b]$. The r^{th} subinterval of P is $\Delta_r = [x_{r-1}, x_r]$, and the length of the r^{th} subinterval is $\delta_r = x_r - x_{r-1}$, $r = 1, 2, 3, \ldots, n$.]

(ii) Norm of a partition :-

Let, $P = \{a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b\}$ be a partition of $[a, b]$. The norm of the partition P is denoted by $\|P\|$ or $\mu(P)$ and defined by $\|P\| = \max\limits_{1 \le r \le n} \delta_r$, $\delta_r = x_r - x_{r-1}$. Norm of the partition is also known as mesh.

For example, the norm of the partition $P = \{0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1\}$ of the interval $[0, 1]$ is $\|P\| = \max\{\frac{1}{4}, \frac{1}{12}, \frac{1}{6}, \frac{1}{2}\}$

$= \frac{1}{2}$

(iii) Upper and lower Darboux sum :-

Let, $f(x)$ be bounded on $[a, b]$.

Let, $P = \{a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b\}$ be a partition of $[a, b]$. Let, $M_r = \sup\limits_{x \in \Delta_r}{}^{m} f(x)$, $m_r = \inf\limits_{x \in \Delta_r}{}^{m} f(x)$, $r = 1, 2, \dots, n$, where $\Delta_r = [x_{r-1}, x_r]$ and $\delta_r = x_r - x_{r-1}$.

The upper Darboux sum of the function $f(x)$ for the partition P of $[a, b]$ is denoted by $U(P, f)$ and defined by

$$U(P, f) = \sum_{r=1}^{n} M_r \delta_r.$$

The lower Darboux sum of $f(x)$ for the partition P of $[a,b]$ is denoted by $L(P,f)$ and defined by
$$L(P,f) = \sum_{r=1}^{n} m_r \delta_r .$$

(ii) <u>Theorem:</u>— Let, $f(x)$ be bounded on $[a,b]$ and P be a partition of $[a,b]$. Then $m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$ where m and M are the bounds of $f(x)$ on $[a,b]$.

<u>Proof:</u>— Let, P be the partition;
$$P : \{a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b\}.$$

Let, $M_r = \sup_{x \in \Delta_r}^{m} f(x)$, $m_r = \inf_{x \in \Delta_r}^{m} f(x)$

$\Delta_r = [x_{r-1}, x_r]$, $\delta_r = x_r - x_{r-1}$; $r = 1, 2, \ldots, n$.

Then we have;
$$m \le m_r \le M_r \le M$$
$$\therefore m\,\delta_r \le m_r \delta_r \le M_r \delta_r \le M \delta_r$$
$$\Rightarrow \sum_{r=1}^{n} m\,\delta_r \le \sum_{r=1}^{n} m_r \delta_r \le \sum_{r=1}^{n} M_r \delta_r \le \sum_{r=1}^{n} M \delta_r$$
$$\Rightarrow m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$$

(iii) <u>lower and upper integral:</u>—

The lower integral of $f(x)$ on $[a,b]$ is denoted by $\underline{\int_a^b} f(x)\,dx$ and defined by $\underline{\int_a^b} f(x)\,dx = \sup^{m} \{L(P,f) : P \in P[a,b]\}$

The ~~low~~ upper integral of $f(x)$ on $[a,b]$ is denoted by $\overline{\int_a^b} f(x)\,dx$ and defined by

$$\overline{\int_a^b} f(x)\,dx = \inf^m \{U(P,f) : P \in P[a,b]\}.$$

(iii) **Riemann-integrable / R-integrable :-**

$f(x)$ is said to be Riemann-integrable or R-integrable on $[a,b]$ if

$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$ and is denoted by

$f(x) \in R[a,b]$.

If $f(x) \in R[a,b]$, then

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx.$$

[Note :- we have,

$$m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$$

$$m(b-a) \le \sup^m L(P,f) \le \inf^m U(P,f) \le M(b-a)$$

$$m(b-a) \le \underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx \le M(b-a)\text{]}$$

1) Show that every constant function is R-integrable on $[a,b]$, i.e. if $f(x) = K$, $\forall x \in [a,b]$, then $f(x) \in R[a,b]$.

$\Rightarrow$ Let, $P = \{a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b\}$ be any partition of $[a,b]$.

Here we have,

$$M = \sup_{x \in [a,b]}^{m} f(x) = K \quad , \quad M_r = \sup_{x \in \Delta_r}^{m} f(x) = K$$

$$m = \inf_{x \in [a,b]}^{m} f(x) = K \quad , \quad m_r = \inf_{x \in \Delta_r}^{m} f(x) = K$$

where $\Delta_r = [x_{r-1}, x_r]$

$\delta_r = x_r - x_{r-1}$,

~~where~~ $r = 1, 2, \dots, n$.

$$\therefore L(P,f) = \sum_{r=1}^{n} m_r \delta_r = \sum_{r=1}^{n} K \delta_r = K(b-a)$$

$$U(P,f) = \sum_{r=1}^{n} M_r \delta_r = \sum_{r=1}^{n} K \delta_r = K(b-a)$$

$$\therefore \underline{\int_a^b} f(x)\,dx = \sup^{m}\{L(P,f) : P \in P[a,b]\}$$

$$= \sup^{m}\{K(b-a) : P \in P[a,b]\}$$

$$= K(b-a)$$

$$\overline{\int_a^b} f(x)\,dx = \inf^{m}\{U(P,f) : P \in P[a,b]\}$$

$$= \inf^{m}\{K(b-a) : P \in P[a,b]\}$$

$$= K(b-a)$$

Since, $\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$

$\therefore f(x) \in R[a,b]$.

$\therefore \int_a^b f(x)\,dx = K(b-a)$.

2) Show that the function $f(x) = \begin{cases} -1, & x \text{ is rational} \\ 1, & x \text{ is irrational} \end{cases}$ is not R-integrable in any interval $[a,b]$.

$\Rightarrow$ Let, $P = \{a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b\}$ be a Partition of $[a,b]$.

$\therefore$ we have,

$\Delta_r = [x_{r-1}, x_r]$, $\delta_r = x_r - x_{r-1}$

$M_r = \sup_{x \in \Delta_r} f(x)$, $m_r = \inf_{x \in \Delta_r} f(x)$, $r = 1, 2, \dots, n$

$= 1$, $= -1$

$\therefore L(P,f) = \sum_{r=1}^{n} m_r \delta_r = \sum_{r=1}^{n} (-1)\delta_r = -(b-a)$

$\therefore U(P,f) = \sum_{r=1}^{n} M_r \delta_r = \sum_{r=1}^{n} 1 \cdot \delta_r = b-a$

$\therefore \underline{\int_a^b} f(x)\,dx = \sup\{L(P,f) : P \in P[a,b]\}$

$= -(b-a)$

$\overline{\int_a^b} f(x)\,dx = \inf\{U(P,f) : P \in P[a,b]\}$

$= b-a$

Since, $\underline{\int_a^b} f(x)\,dx \neq \overline{\int_a^b} f(x)\,dx$

$\therefore f(x)$ is not integrable.

3) Show that $f(x) = x^2$ is integrable on ~~[a,b]~~ $[0,1]$.

$\Rightarrow$ Let, $P = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n} = 1\}$ be any partition of $[0,1]$.

Here, $\Delta_r = [\frac{r-1}{n}, \frac{r}{n}]$, $\delta_r = \frac{1}{n}$

Since, $f(x)$ is monotonic increasing in each subinterval Δ_r,

$$M_r = f\left(\frac{r}{n}\right) = \frac{r^2}{n^2}, \quad r = 1, 2, \ldots, n.$$

$$m_r = f\left(\frac{r-1}{n}\right) = \frac{(r-1)^2}{n^2}$$

$$\therefore L(P,f) = \sum_{r=1}^{n} m_r \delta_r = \sum_{r=1}^{n} \frac{(r-1)^2}{n^2} \cdot \frac{1}{n}$$

$$= \frac{1}{n^3}[0^2 + 1^2 + 2^2 + \ldots + (n-1)^2]$$

$$= \frac{1}{n^3} \frac{(n-1)n(2n-1)}{6}$$

$$= \frac{1}{6}(1 - 1/n)(2 - 1/n)$$

$$\therefore U(P,f) = \sum_{r=1}^{n} M_r \delta_r = \sum_{r=1}^{n} \frac{r^2}{n^2} \cdot \frac{1}{n}$$

$$= \frac{1}{n^3}[1^2 + 2^2 + \ldots + n^2]$$

$$= \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{1}{6}(1 + 1/n)(2 + 1/n)$$

$$\therefore \overline{\int_0^1} f(x)\,dx = \inf{}^m \{U(P,f) : P \in P[0,1]\}$$

$$= \inf{}^m \left\{ \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) : n \in \mathbb{N} \right\}$$

$$= \lim_{n \to \infty} \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

$$= \frac{1}{3}$$

$$\therefore \underline{\int_0^1} f(x)\,dx = \sup{}^m \{L(P,f) : P \in P[0,1]\}$$

$$= \sup{}^m \left\{ \frac{1}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right) : n \in \mathbb{N} \right\}$$

$$= \lim_{n \to \infty} \frac{1}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)$$

$$= \frac{1}{3}$$

Since, $\underline{\int_0^1} f(x)\,dx = \overline{\int_0^1} f(x)\,dx$

$\therefore f(x) = x^2$ is integrable.

$\therefore \int_0^1 f(x)\,dx = \frac{1}{3}$

Show that $f(x) = x^2$ is integrable on $[0, k]$.

4) Show that $f(x) = x^2$ is integrable on $[a, b]$.

5) Show that the function $f(x) = x^2$ is

Refinement :- Let, P be a partition of an interval [a, b]. A Partition P^* is said to be refinement of the ~~partion~~ Partition P if $P^* \supseteq P$.

If P^* be a refinement of P then P^* contains all the points of P and some more points.

Theorem :- Let, P^* be a refinement of the partition P of [a, b] and f(x) be bounded on [a, b]. Then —

i) $L(P,f) \leq L(P^*,f)$

ii) $U(P^*,f) \leq U(P,f)$

[**Note :-** Let, P be a Partition of [a, b] with $\mu(P) < \delta$ and $|f(x)| \leq K$, $\forall x \in [a,b]$. Let, P^* be a refinement of P containing p more points, then —

i) $L(P,f) \leq L(P^*,f) \leq L(P,f) + 2pK\delta$

ii) $U(P,f) - 2pK\delta \leq U(P^*,f) \leq U(P,f)$]

Darboux theorem :- Let, f(x) be bounded on [a, b]. Then to each $\epsilon > 0$ there corresponds $\delta > 0$ such that

i) $U(P,f) < \overline{\int_a^b} f(x)\,dx + \epsilon$

ii) $L(P,f) > \underline{\int_a^b} f(x)\,dx - \epsilon$, for every partition P of [a, b] with $\|P\| < \delta$.

Proof :-

Since, $\underline{\int_a^b} f(x)\,dx$ is the supremum of the set $\{L(P,f) : P \in P[a,b]\}$.

$\therefore \exists$ a partition P_1 of $[a,b]$ such that

$$L(P_1,f) > \underline{\int_a^b} f(x)\,dx - \frac{\epsilon}{2} \text{ ——— (i)}$$

Let, $P_1 = \{a = x_0 < x_1 < x_2 < \dots < x_p = b\}$ be the partition.

Similarly,

$\exists$ a partition P_2 of $[a,b]$ such that

$$U(P_2,f) < \overline{\int_a^b} f(x)\,dx + \frac{\epsilon}{2} \text{ ——— (ii)}$$

Let, $P_2 = \{a = x_0' < x_1' < x_2' < \dots < x_{p'}' = b\}$

Since, $f(x)$ is bounded on $[a,b]$,

$\exists$ a positive real number K such that

$|f(x)| \le K, \ \forall x \in [a,b]$.

Let, $2(p'-1)K\delta = \frac{\epsilon}{2}$ and P be a partition of $[a,b]$ with $\mu(P) < \delta$,

Let, P^* be a refinement of $[a,b]$ such that $P^* = P \cup P_2$ [common refinement of P and P_2]

Since, P^* contains atmost $(p'-1)$ points more.

$\therefore$ we have,

$$U(P,f) - 2(p'-1)K\delta \le U(P^*,f) \le U(P_2,f)$$

$$< \overline{\int_a^b} f(x)\,dx + \frac{\epsilon}{2} \quad \text{(by (ii))}$$

$$U(P,f) < \overline{\int_a^b} f(x)\,dx + \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$$\Rightarrow U(P,f) < \overline{\int_a^b} f(x)\,dx + \epsilon$$

Similarly, we can show that $L(P,f) > \underline{\int_a^b} f(x)\,dx - \epsilon$.

(III) Condition of integrability :-

A bounded function $f(x)$ on $[a,b]$ is integrable iff to each $\epsilon > 0$, $\exists$ a $\delta > 0$ such that $U(P,f) - L(P,f) < \epsilon$ for every partition P with $\mu(P) < \delta$.

Proof :-

Let us first suppose that $f(x)$ is integrable on $[a,b]$

$$\therefore \underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx. \quad \text{—— (i)}$$

Since, $\mu(P) < \delta$

$\therefore$ By Darboux theorem,

$$U(P,f) < \overline{\int_a^b} f(x)\,dx + \epsilon/2 \quad \text{—— (ii)}$$

$$\text{and } L(P,f) > \underline{\int_a^b} f(x)\,dx - \epsilon/2 \quad \text{—— (iii)}$$

$$\cancel{\Rightarrow L(P,f) < -\underline{\int_a^b} f(x)\,dx + \epsilon}$$

from (i), (ii) and (iii),

$$U(P,f) - L(P,f) < \epsilon.$$

$\therefore$ The condition is necessary.

Conversely suppose that $U(P,f) - L(P,f) < \epsilon$, where $\mu(P) < \delta$.

We have,

$$L(P,f) \leq \underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx \leq U(P,f)$$

$$\therefore U(P,f) - L(P,f) \geq \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx$$

$$\therefore \overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx < \epsilon$$

$$\therefore \overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx \quad [\because \epsilon > 0 \text{ being arbitrary}]$$

$\therefore f(x)$ is integrable.

$\therefore$ The condition is sufficient.

4) Q. Show that $f(x)=x^2$ is integrable on $[0,k]$.

=> Let, $P=\{0, \frac{k\cdot 1}{n}, \frac{k\cdot 2}{n}, \frac{k\cdot 3}{n}, \ldots, \frac{k\cdot n}{n}\}$ be the partition of $[0,k]$

Here, $\Delta_r = \left[\frac{k(r-1)}{n}, \frac{kr}{n}\right]$, $\delta_r = \frac{k}{n}$, $r=1,2,\ldots,n$

Since, $f(x)$ is increasing in each subinterval, Δ_r

$\therefore M_r = \sup\limits_{x\in\Delta_r}{}^m f(x) = f\left(\frac{kr}{n}\right) = \frac{k^2r^2}{n^2}$

$\therefore m_r = \inf\limits_{x\in\Delta_r}{}^m f(x) = f\left(\frac{k(r-1)}{n}\right) = \frac{k^2(r-1)^2}{n^2}$

$\therefore L(P,f) = \sum\limits_{r=1}^{n} m_r\delta_r = \sum\limits_{r=1}^{n} \frac{k^2(r-1)^2}{n^2}\cdot\frac{k}{n} = \frac{k^3}{n^3}\left[0^2+1^2+\ldots+(n-1)^2\right]$

$= \frac{k^3}{n^3}\,\frac{(n-1)n(2n-1)}{6} = \frac{k^3}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)$

$\therefore U(P,f) = \sum\limits_{r=1}^{n} M_r\delta_r = \sum\limits_{r=1}^{n}\frac{k^2r^2}{n^2}\cdot\frac{k}{n} = \frac{k^3}{n^3}\,\frac{n(n+1)(2n+1)}{6}$

$= \frac{k^3}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$

$\therefore \underline{\int_0^k} f(x)\,dx = \sup{}^m\{L(P,f): P\in P[0,k]\} = \sup{}^m\left\{\frac{k^3}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right): n\in\mathbb{N}\right\}$

$= \lim\limits_{n\to\infty}\frac{k^3}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right) = \frac{k^3}{3}$

$\therefore \overline{\int_0^k} f(x)\,dx = \inf{}^m\{U(P,f): P\in P[0,k]\} = \inf{}^m\left\{\frac{k^3}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right): n\in\mathbb{N}\right\}$

$= \lim\limits_{n\to\infty}\frac{k^3}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) = \frac{k^3}{3}$

$\because \underline{\int_0^k} f(x)\,dx = \overline{\int_0^k} f(x)\,dx$ and $\int_0^k f(x)\,dx = \frac{k^3}{3}$

$\therefore f(x)=x^2$ is integrable on $[0,k]$

5) Show that the function $f(x)=x^2$ is integrable on $[a,b]$

=> Let, $P=\{a, a+h, a+2h, \ldots, a+nh=b\}$ be the partition of $[a,b]$.

Here, $\Delta_r = [a+\overline{r-1}h, a+rh]$, $\delta_r = h$, $r=1,2,\ldots,n$

Since, $f(x)$ is increasing in each subinterval Δ_r,

$$\therefore M_r = \sup_{x\in\Delta_r}{}^{m} f(x) = f(a+rh) = (a+rh)^2$$

$$\therefore m_r = \inf_{x\in\Delta_r}{}^{m} f(x) = f(a+\overline{r-1}h) = (a+\overline{r-1}h)^2$$

$$\therefore U(P,f) = \sum_{r=1}^{n} M_r \delta_r$$

$$= \sum_{r=1}^{n} (a+rh)^2 h = \sum_{r=1}^{n} (a^2h + r^2h^3 + 2arh^2)$$

$$= a^2hn + h^3\frac{n(n+1)(2n+1)}{6} + 2ah^2\frac{n(n+1)}{2}$$

$$= a^2(b-a) + \frac{n^3h^3}{6}(1+1/n)(2+1/n) + a n^2h^2(1+1/n)$$

$$= a^2(b-a) + \frac{(b-a)^3}{6}(1+1/n)(2+1/n) + a(b-a)^2(1+1/n)$$

$$\therefore L(P,f) = \sum_{r=1}^{n} m_r \delta_r$$

$$= \sum_{r=1}^{n} (a+\overline{r-1}h)^2 h$$

$$= \sum_{r=1}^{n} \{a^2h + \overline{r-1}^2h^3 + 2ah^2(r-1)\}$$

$$= a^2hn + h^3[0^2+1^2+\ldots+(n-1)^2] + 2ah^2[0+1+\ldots+n-1]$$

$$= a^2(b-a) + h^3\frac{(n-1)n(2n-1)}{6} + 2ah^2\frac{(n-1)n}{2}$$

$$= a^2(b-a) + \frac{n^3h^3}{6}(1-1/n)(2-1/n) + a n^2h^2(1-1/n)$$

$$= a^2(b-a) + \frac{(b-a)^3}{6}(1-1/n)(2-1/n) + a(b-a)^2(1-1/n)$$

$$\therefore \underline{\int_a^b} f(x)dx = \sup^m\{L(P,f): P \in P[a,b]\} = \sup^m\{L(P,f): n \in N\}$$

$$= \lim_{n\to\infty}\left[a^2(b-a) + \frac{(b-a)^3}{6}(1-1/n)(2-1/n) + a(b-a)^2(1-1/n)\right]$$

$$= a^2(b-a) + \frac{(b-a)^3}{3} + a(b-a)^2$$

$$\therefore \overline{\int_a^b} f(x)dx = \inf^m\{U(P,f): P \in P[a,b]\} = \inf^m\{U(P,f): n \in N\}$$

$$= \lim_{n\to\infty}\left[a^2(b-a) + \frac{(b-a)^3}{6}(1+1/n)(2+1/n) + a(b-a)^2(1+1/n)\right]$$

$$= a^2(b-a) + \frac{(b-a)^3}{3} + a(b-a)^2$$

$$\because \underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx$$

$\therefore f(x)$ is integrable on $[a,b]$ and $\int_a^b f(x)dx = a^2(b-a) + \frac{(b-a)^3}{3} + a(b-a)^2$

$$= a^2(b-a) + (b-a)^2\left[a + \frac{b-a}{3}\right]$$

$$= a^2(b-a) + (b-a)^2\,\frac{2a+b}{3}$$

$$= (b-a)\left(a^2 + \frac{2ab+b^2-2a^2-ab}{3}\right)$$

$$= (b-a)\,\frac{a^2+ab+b^2}{3}$$

$$= \frac{b^3-a^3}{3}$$

Theorem :- A bounded function $f(x)$ on $[a,b]$ is integrable iff for $\epsilon > 0$, $\exists$ a partition P of $[a,b]$ such that $U(P,f) - L(P,f) < \epsilon$.

Proof :-

Let us first suppose that $f(x)$ is integrable.

$$\therefore \underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx \quad \text{--- (i)}$$

Since, $\overline{\int_a^b} f(x)\,dx$ and $\underline{\int_a^b} f(x)\,dx$ are the infimum and supremum of the sets of upper sums and lower sums respectively, corresponding to $\epsilon > 0$ $\exists$ Partitions P_1, P_2 of $[a,b]$ such that

$$U(P_1,f) < \overline{\int_a^b} f(x)\,dx + \epsilon/2 \quad \text{--- (ii)}$$

$$L(P_2,f) > \underline{\int_a^b} f(x)\,dx - \epsilon/2 \quad \text{--- (iii)}$$

Let, $P = P_1 \cup P_2$ (i.e. P is the common refinement of P_1 and P_2).

$\therefore$ We have,

$$U(P,f) \leq U(P_1,f) < \overline{\int_a^b} f(x)\,dx + \epsilon/2$$

$$= \underline{\int_a^b} f(x)\,dx + \epsilon/2 \quad [\text{by (i)}]$$

$$< L(P_2,f) + \epsilon \quad (\text{by (iii)})$$

$$\leq L(P,f) + \epsilon$$

$U(P,f) - L(P,f) < \epsilon$

Thus the condition is necessary.

Conversely suppose that, corresponding to $\epsilon > 0$, $\exists$ a partition P of $[a,b]$ such that $U(P,f) - L(P,f) < \epsilon$.

We have, for the partition P,

$$L(P,f) \le \underline{\int_a^b} f(x)dx \le \overline{\int_a^b} f(x)dx \le U(P,f)$$

$$\therefore \overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx \le U(P,f) - L(P,f) < \epsilon$$

$$\therefore \left| \overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx \right| < \epsilon$$

$$\Rightarrow \overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx \quad [\because \epsilon > 0 \text{ is arbitrary}]$$

(VIII) **Integral as limit of sums : Riemann sums :—**

Let, $P = \{a = x_0 < x_1 < x_2 < \dots < x_{r-1} < x_r < \dots < x_{n-1} < x_n = b\}$ be a partition of $[a,b]$.

Let, ξ_r be any arbitrary point in the rth subinterval $\Delta_r = [x_{r-1}, x_r]$ whose length is $\delta_r = x_r - x_{r-1}$, $r = 1, 2, \dots, n$.

Let, $S(P,f) = \sum_{r=1}^{n} f(\xi_r)\,\delta_r$.

This sum is known as Riemann sum.

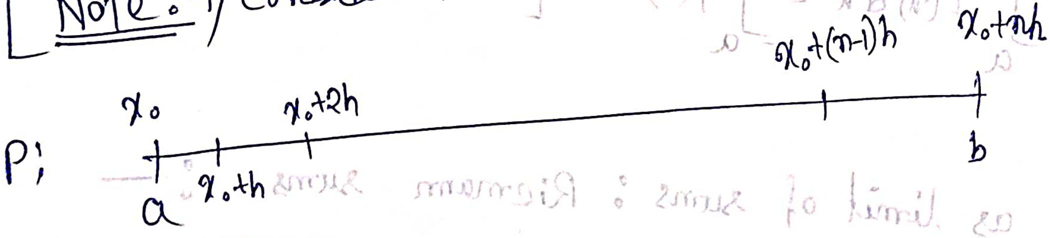
$\lim_{\mu(P)\to 0} S(P,f) = A$ implies that corresponding to $\epsilon > 0$, $\exists$ a positive δ such that $|S(P,f) - A| < \epsilon$ for those partitions P with $\mu(P) < \delta$.

If $\lim_{\mu(P)\to 0} S(P,f)$ exists, then $f(x)$ is integrable and $\lim_{\mu(P)\to 0} S(P,f) = \int_a^b f(x)\,dx$.

[Note :- $\mu(P) \to 0$ also implies $n\to\infty$.

$\therefore \lim_{\mu(P)\to 0} S(P,f) = \lim_{n\to\infty} S(P,f)$.]

[Note :- i) Consider the partition P:



Here, $\mu(P) = h = \dfrac{b-a}{n}$

and $x_r = x_0 + rh$, $r = 1, 2, \ldots, n$

choosing $\xi_r = x_r$ we have,

$$\lim_{n\to\infty} \frac{b-a}{n} \sum_{r=1}^{n} f\left(x_0 + \frac{r(b-a)}{n}\right) = \int_a^b f(x)\,dx.$$

ii) $$\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)\,dx.$$

I(a) Evaluate the limit sum: $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{n}{n^2+r^2}$

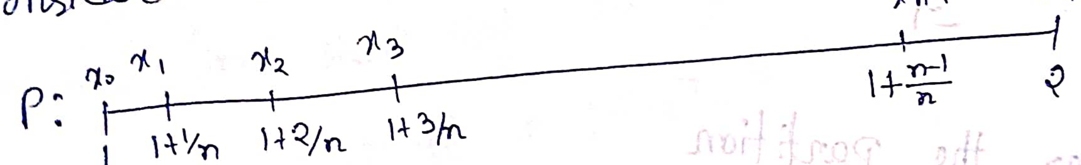
$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{n}{n^2+r^2}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{n^2}{n^2+r^2}=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{1+(r/n)^2}$$

$$=\int_0^1\frac{dx}{1+x^2}=\left[\tan^{-1}x\right]_0^1=\frac{\pi}{4}$$

4) Show that $\int_1^2 f(x)\,dx=\frac{11}{2}$, where $f(x)=3x+1$.

$\Rightarrow$ Consider the partition



Here, $x_r=1+\frac{r}{n}$, $\delta_r=\frac{1}{n}$, $\mu(P)=\frac{1}{n}$

$r=1,2,\dots,n$

Let us choose, $\xi_r=x_r$,

$\therefore$ The Riemann sum, $S(P,f)=\sum_{r=1}^{n}f(\xi_r)\delta_r$

$$=\sum_{r=1}^{n}f(x_r)\frac{1}{n}$$

$$=\sum_{r=1}^{n}(3x_r+1)\frac{1}{n}$$

$$=\frac{1}{n}\sum_{r=1}^{n}3\left(1+\frac{r}{n}\right)+1$$

$$=\frac{1}{n}\left(\sum_{r=1}^{n}\right)4+3\frac{r}{n}$$

$$=\frac{1}{n}\sum_{r=1}^{n}4+\frac{3}{n^2}\sum_{r=1}^{n}r$$

$$= \frac{1}{n}\cdot 4n + \frac{3}{n^2}\cdot\frac{n(n+1)}{2}$$

$$= 4 + \frac{3}{2} + \frac{3}{2n}$$

$$= \frac{11}{2} + \frac{3}{2n}$$

$$\therefore \int_1^2 f(x)\,dx = \lim_{\mu(P)\to 0} S(P,f) = \lim_{n\to\infty} S(P,f)$$

$$= \lim_{n\to\infty}\left(\frac{11}{2} + \frac{3}{2n}\right)$$

$$= \frac{11}{2}$$

5) Evaluate $\int_{-1}^{1} |x|\,dx$.

$\Rightarrow$ Consider the partition

$$P: \begin{array}{ccccccccc} x_0 & x_1 & x_2 & x_{n-1} & x_n & y_1 & y_2 & y_{n-1} & y_n \\ -1 & -1+\frac{1}{n} & -1+\frac{2}{n} & -1+\frac{n-1}{n} & -1+\frac{n}{n} = 0 & -1+\frac{n+1}{n} & -1+\frac{n+2}{n} & -1+\frac{2(n-1)}{n} & -1+\frac{2n}{n} \end{array}$$

Here, $\|P\| = \frac{1}{n}$, $\delta_r = \frac{1}{n}$, $\delta_r' = \frac{1}{n}$,

$x_r = -1 + \frac{r}{n}$, $y_r = \frac{r}{n}$.

Let us choose, $\xi_r = x_r$, $\xi_r' = y_r$

We have, $f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

$\therefore$ Riemann sum, $S(P,f) = \sum_{r=1}^{n} f(\xi_r)\delta_r + \sum_{r=1}^{n} f(\xi_r')\delta_r'$

$$= \sum_{r=1}^{n} f(-1+r/n)\frac{1}{n} + \sum_{r=1}^{n} f\left(\frac{r}{n}\right)\frac{1}{n}$$

$$= \frac{1}{n}\sum_{r=1}^{n}(1-r/n) + \frac{1}{n}\sum_{r=1}^{n}\frac{r}{n}$$

$$= \frac{1}{n}\sum_{r=1}^{n}(1) = \frac{1}{n}\cdot n = 1$$

$$\therefore \int_{-1}^{1}|x|\,dx = \lim_{\mu(P)\to 0} S(P,f) = \lim_{n\to\infty} S(P,f)$$

$$= \lim_{n\to\infty} 1 = 1.$$

(III) **Equivalence of two definitions :-**

We shall first show that $\mathrm{Def}^n 1 \Rightarrow \mathrm{Def}^n 2$

Let, $f(x)$ be integrable on $[a,b]$

$\therefore$ By first definition we have,

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx \quad \text{—— (i)}$$

Let, $\epsilon > 0$

$\therefore$ By Darboux theorem, corresponding to $\epsilon>0$, $\exists$ a $\delta > 0$ such that

$$U(P,f) < \overline{\int_a^b} f(x)\,dx + \epsilon = \int_a^b f(x)\,dx + \epsilon \quad \text{—— (ii)}$$

$$L(P,f) > \underline{\int_a^b} f(x)\,dx - \epsilon = \int_a^b f(x)\,dx - \epsilon \quad \text{—— (iii)}$$

with $\mu(P) < \delta$

Let, $P = \{a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b\}$ be a partition of $[a,b]$ with $\|P\| < \delta$.

We have,

$$m_r \le f(\xi_r) \le M_r$$

$$\Rightarrow m_r \delta_r \le f(\xi_r)\delta_r \le M_r \delta_r$$

$$\Rightarrow \sum_{r=1}^{n} m_r \delta_r \le \sum_{r=1}^{n} f(\xi_r)\delta_r \le \sum_{r=1}^{n} M_r \delta_r$$

$$\Rightarrow L(P,f) \le S(P,f) \le U(P,f)$$

$$\Rightarrow \int_a^b f(x)dx - \epsilon < L(P,f) \le S(P,f) \le U(P,f) < \int_a^b f(x)dx + \epsilon$$

[by (ii) and (iii)]

$$\Rightarrow \int_a^b f(x)dx - \epsilon < S(P,f) < \int_a^b f(x)dx + \epsilon$$

$$\Rightarrow \left| S(P,f) - \int_a^b f(x)dx \right| < \epsilon,$$ for partitions P of $[a,b]$ with $\mu(P) < \delta$

$$\Rightarrow \lim_{n\to\infty} S(P,f) = \int_a^b f(x)dx$$

$\therefore$ first definition $\Rightarrow$ second definition.

Next we shall show that,

Def^n 1 .

[Def^n 2 $\Rightarrow$

Let, $\lim_{\mu(P)\to 0} S(P,f) = \int_a^b f(x)dx = A$ (say).

$\therefore$ Corresponding to $\epsilon > 0$, $\exists$ a $\delta > 0$ such that

$|S(P,f) - A| < \epsilon/2$ with $\mu(P) < \delta$.

$\because$ the limit exists, $f(x)$ is bounded on $[a,b]$.

$\therefore$ from the above relation,

$$A - \epsilon/2 < S(P,f) < A + \epsilon/2 \quad \text{with } \mu(P) < \delta. \quad \text{———(i)}$$

from (i) we have,

$$A - \epsilon/2 < L(P,f) \leq U(P,f) < A + \epsilon/2$$

$$\Rightarrow \quad U(P,f) - L(P,f) < \epsilon$$

$$\therefore \underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx.$$

$\therefore$ 2nd definition $\Rightarrow$ 1st definition.

(III) <u>Theorem</u> :- Let, $f(x)$ be monotone on $[a,b]$. Then $f(x)$ be integrable.

<u>Proof</u> :-

<u>Case-I</u> :-

Let, $f(x)$ be monotonic ~~inc~~ increasing on $[a,b]$.

$\forall x \in [a,b]$ we have,

$$a \leq x \leq b$$

$$f(a) \leq f(x) \leq f(b)$$

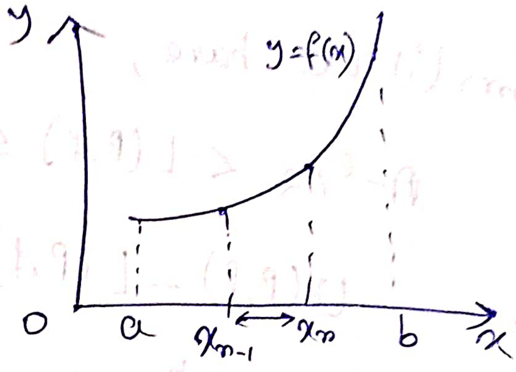
$\therefore f(x)$ is bounded on $[a,b]$.

Let, $\epsilon > 0$ be arbitrary and $\delta = \dfrac{\epsilon}{f(b) - f(a) + 1}$.

Let, $P = \{a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b\}$ be a partition of $[a,b]$ with $\|P\| < \delta$.

Let, $\Delta_r = [x_{r-1}, x_r]$, $\delta_r = x_r - x_{r-1}$

$M_r = \sup\limits_{x \in \Delta_r}^{m} f(x)$, $m_r = \inf\limits_{x \in \Delta_r}^{m} f(x)$, $r = 1, 2, \ldots, n$.



Since, $f(x)$ is ~~mont~~ monotonic increasing on $[a, b]$, it is mondotonic increasing on Δ_r.

$\therefore m_r = f(x_{r-1})$

$M_r = f(x_r)$

$$\therefore U(P,f) - L(P,f) = \sum_{r=1}^{n} (M_r - m_r)\delta_r$$

$$\leq \sum_{r=1}^{n} (M_r - m_r)\|P\| \quad [\because \delta_r \leq \|P\|]$$

$$= \|P\| \sum_{r=1}^{n} (M_r - m_r)$$

$$= \|P\| \sum_{r=1}^{n} \left(f(x_r) - f(x_{r-1})\right)$$

$$< \delta\left(f(b) - f(a)\right) \quad [\because \|P\| < \delta]$$

$$= \frac{\epsilon}{f(b) - f(a) + 1}\left(f(b) - f(a)\right)$$

$$< \epsilon$$

$\therefore f \in R[a, b]$.

Case - II :- Let, $f(x)$ be monotonic decreasing.

Similarly, we can show that $f(x) \in R[a,b]$.

$\therefore$ Combining cases I and II, $f(x) \in R[a, b]$.

(iii) **Theorem:-** Let, $f(x)$ be continuous on $[a,b]$. Then $f(x) \in R[a,b]$.

Proof:- Since, $f(x)$ is continuous on $[a,b]$,

it is uniformly continuous on $[a,b]$.

$\therefore$ Corresponding to $\epsilon > 0$, $\exists\ \delta > 0$ such that

$$|x' - x''| < \delta \Rightarrow |f(x') - f(x'')| < \frac{\epsilon}{b-a}, \ \forall x', x'' \in [a,b]. \quad \text{(i)}$$

Let, $P = \{a = x_0 < x_1 < x_2 < \dots < x_{r-1} < x_n = b\}$ be a partition of $[a,b]$

with $\|P\| < \delta$.

Let, $\Delta_r = [x_{r-1}, x_r]$; $\delta_r = x_r - x_{r-1}$

$M_r = \sup\limits_{x \in \Delta_r} f(x)$, $m_r = \inf\limits_{x \in \Delta_r} f(x)$, $r = 1, 2, \dots, n$.

Since, $f(x)$ is continuous on $[a,b]$, it is continuous on Δ_r,

$r = 1, 2, \dots, n$.

$\therefore$ By attainment property of a continuous function on a closed interval, $\exists\ \alpha_r, \beta_r \in \Delta_r$ such that

$$f(\alpha_r) = m_r, \quad f(\beta_r) = M_r, \ r = 1, 2, \dots, n.$$

$\therefore$ We have,

$$U(P,f) - L(P,f) = \sum_{r=1}^{n} (M_r - m_r)\, \delta_r$$

$$= \sum_{r=1}^{n} |M_r - m_r|\, \delta_r$$

$$= \sum_{r=1}^{n} |f(\beta_r) - f(\alpha_r)|\, \delta_r$$

by (i) $[\because |\beta_r - \alpha_r| \le \delta_r < \delta]$

$$< \sum_{r=1}^{n} \frac{\epsilon}{b-a} \delta_r$$

$$= \frac{\epsilon}{b-a} \sum_{r=1}^{n} \delta_r$$

$$= \frac{\epsilon}{b-a} (b-a)$$

$$= \epsilon$$

$\therefore U(P,f) - L(P,f) < \epsilon$ with $\|P\| < \delta$.

$\therefore f(x) \in R[a,b]$.

[Note:– $f(x) \in C[a,b] \Rightarrow f(x) \in R[a,b]$]

Oscillatory sum:– Let, P be a partition of $[a,b]$.
Then $U(P,f) - L(P,f) = \sum_{r=1}^{n} (M_r - m_r)\delta_r$ is called the oscillatory sum of $f(x)$ on $[a,b]$ for the partition P.

$(M_r - m_r)$ is called the oscillation of $f(x)$ on rth subinterval Δ_r.

Theorem:– Let, $f(x)$ be continuous on $[a,b]$ except for a finite number of points, then $f(x) \in R[a,b]$, provided $f(x)$ is bounded.

Proof:–
Since, $f(x)$ is bounded on $[a,b]$,
$\exists\ B > 0$ such that $|f(x)| \leq B,\ \forall x \in [a,b]$ ——— (i)

Let, $\epsilon > 0$ be arbitrary.

Let, $f(x)$ be discontinuous at $x_1, x_2, \dots, x_m$.

Case - I :-

Let, $x_1 \neq a$ and $x_m \neq b$ and $a < x_1 < x_2 < \dots < x_m < b$.

Let us consider, m non-overlapping intervals

$$\Delta_r = \left[x_r - \frac{\delta_r}{2}, x_r + \frac{\delta_r}{2} \right], \quad r = 1, 2, \dots, m.$$

Where $\delta_1 + \delta_2 + \dots + \delta_m < \frac{\epsilon}{4B}$ —— (ii)

Let, $M_r = \sup\limits_{x \in \Delta_r}^{m} f(x)$, $m_r = \inf\limits_{x \in \Delta_r}^{m} f(x)$, $r = 1, 2, \dots, m$

Clearly, $f(x)$ is continuous on each subinterval

$\left[a, x_1 - \frac{\delta_1}{2}\right]$, $\left[x_1 + \frac{\delta_1}{2}, x_2 - \frac{\delta_2}{2}\right]$, $\dots$, $\left[x_m + \frac{\delta_m}{2}, b\right]$

$\therefore f(x)$ is ~~integrable~~ continuous on each of ~~the~~ these subintervals and hence integrable.

$\therefore \exists$ partitions $P_1, P_2, \dots, P_{m+1}$ such that

$$U(P_i, f) - L(P_i, f) < \frac{\epsilon}{2(m+1)}, \quad i = 1, 2, \dots, m+1 \text{ —— (iii)}$$

Let, $P = P_1 \cup P_2 \cup \dots \cup P_{m+1}$ be a partition of $[a, b]$.

$\therefore$ we have,

$$U(P, f) - L(P, f) = U(P_1, f) - L(P_1, f) + (M_1 - m_1)\delta_1 + U(P_2, f) - L(P_2, f) + (M_2 - m_2)\delta_2 + \dots + (M_m - m_m)\delta_m + U(P_{m+1}, f) - L(P_{m+1}, f)$$

$$= \sum_{i=1}^{m+1} \left\{ U(P_i, f) - L(P_i, f) \right\} + \sum_{r=1}^{m} (M_r - m_r)\delta_r$$

$$< \sum_{i=1}^{m+1} \frac{\epsilon}{2(m+1)} + \sum_{r=1}^{m} 2B\,\delta_r \quad \left[\because M_r - m_r \le 2B \text{ and by (iii)}\right]$$

$$= \frac{\epsilon}{2(m+1)} \cdot (m+1) + 2B \sum_{r=1}^{m} \delta_r$$

$$< \frac{\epsilon}{2} + 2B \cdot \frac{\epsilon}{4B} \quad [\text{by (ii)}]$$

$$= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$\therefore f(x) \in R[a,b]$.

Case - II :-

If $x_1 = a$ or $x_m = b$ or both, proceeding as above we can show that $f(x) \in R[a,b]$.

(III) Theorem :- Let, $f(x)$ be bounded on $[a,b]$ and $f(x)$ has infinite number of points of discontinuity in $[a,b]$. If the set of points of discontinuity of $f(x)$ on $[a,b]$ have a finite number of limit points, then $f(x) \in R[a,b]$.

Proof :-

Since, $f(x)$ is bounded on $[a,b]$, $\exists$ a positive real number B such that

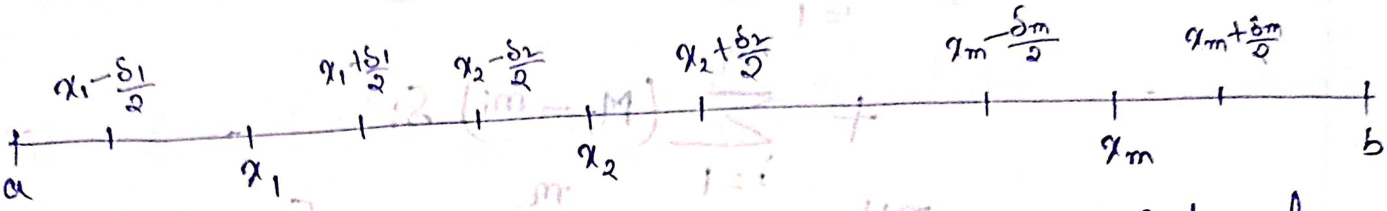
$$|f(x)| \le B, \ \forall x \in [a,b] \quad \text{—— (i)}$$

Let, S be the set of points of discontinuity of $f(x)$ on $[a,b]$.

given that, S is infinite set and S has finite number of limit points.

Let, $x_1, x_2, \ldots, x_m$ be the limit points of S.

Let, $a < x_1 < x_2 < \ldots < x_m < b$



Let us enclose $x_1, x_2, \ldots, x_m$ by m non-overlapping intervals

$\Delta_i = \left[x_i - \frac{\delta_i}{2}, x_i + \frac{\delta_i}{2}\right]$, $i = 1, 2, \ldots, m$

Where $\delta_1 + \delta_2 + \ldots + \delta_m < \frac{\epsilon}{4B}$ and $\epsilon > 0$ be arbitrary. ——— (ii)

Since, $f(x)$ is bounded on $[a, b]$, it is bounded on every subinterval of $[a, b]$.

Let, $M_i = \sup_{x \in \Delta_i} f(x)$, $m_i = \inf_{x \in \Delta_i} f(x)$, $i = 1, 2, \ldots, m$.

Clearly, each subinterval: $\left[a, x_1 - \frac{\delta_1}{2}\right]$, $\left[x_1 + \frac{\delta_1}{2}, x_2 - \frac{\delta_2}{2}\right]$, $\ldots$, $\left[x_m + \frac{\delta_m}{2}, b\right]$ contains finite number of points of discontinuity, Otherwise, S ~~possess~~ possess a new limit point.

$\therefore f(x)$ is integrable on each of these subintervals.

$\therefore \exists$ partitions $P_1, P_2, \ldots, P_{m+1}$ respectively of these subintervals such that $U(P_i, f) - L(P_i, f) < \frac{\epsilon}{2(m+1)}$, $i = 1, 2, \ldots, m+1$ ——— (iii)

Let, $P = P_1 \cup P_2 \cup \ldots \cup P_{m+1}$

$\therefore$ We have,

$$U(P,f)-L(P,f) = \sum_{i=1}^{m+1}\{U(P_i,f)-L(P_i,f)\} + \sum_{i=1}^{m}(M_i-m_i)\,\delta_i$$

$$< \sum_{i=1}^{m+1}\frac{\epsilon}{2(m+1)} + \sum_{i=1}^{m} 2B\delta_i \quad \left[\because M_i - m_i \le 2B\right]$$

$$< \frac{\epsilon}{2(m+1)}\cdot(m+1) + 2B\,\frac{\epsilon}{4B} \quad [\text{by (ii)}]$$

$$= \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon$$

$\therefore f(x) \in R[a,b]$.

If $x_1 = a$ or $x_m = b$ or both, then modifying the subintervals and proceeding as above we can show that $f(x) \in R[a,b]$.

(III) **Theorem :–** Let, $f(x) \in R[a,b]$ and $g(x)$ defined on $[a,b]$ such that $f(x) = g(x)$ except finite number of points of $[a,b]$. Then ~~f(x)~~ $g(x) \in R[a,b]$.

1) Show that $f(x) \in R[0,3]$, where $f(x) = [x]$, $\forall x \in [0,3]$.

$\Rightarrow$ Here,
$$f(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & 2 \le x < 3 \end{cases}$$

Clearly, $f(x)$ is continuous everywhere on $[0,3]$ except at $1, 2$ and 3.

Since, the number of points of discontinuity of $f(x)$ is finite, $f(x) \in R[0,3]$.

2) Let, $f(x) = (-1)^n, \quad \frac{1}{n+1} < x \le \frac{1}{n}, \quad n = 1, 2, 3, \ldots$

$= 0, \quad x = 0$

Then show that $f(x)$ is integrable on $[0,1]$.

⇒ we have,

$$f(x) = 0, \quad x = 0$$
$$= -1, \quad \tfrac{1}{2} < x \le 1$$
$$= 1, \quad \tfrac{1}{3} < x \le \tfrac{1}{2}$$
$$= -1, \quad \tfrac{1}{4} < x \le \tfrac{1}{3}$$

$- - - - - - - - -$

Here, $f(x)$ has discontinuity at $x = 0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

The set of points of discontinuity of f on $[0,1]$ is

$S = \{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$

clearly, S has only one limit point, namely 0.

Since, $f(x)$ has infinite number of points of discontinuity having finite number of limit point, $f(x) \in R[0,1]$.

3) Let, $f(x) = \frac{1}{2^n}, \quad \frac{1}{2^{n+1}} < x \le \frac{1}{2^n}, \quad n = 0, 1, 2, \ldots$

$= 0, \quad x = 0$

Show that $f(x)$ is integrable on $[0,1]$.

=> We have,

$$f(x) = 1, \quad \frac{1}{2} < x \le 1$$
$$= \frac{1}{2}, \quad \frac{1}{2^2} < x \le \frac{1}{2}$$
$$= \frac{1}{2^2}, \quad \frac{1}{2^3} < x \le \frac{1}{2^2}$$
$$\cdots$$

Here, $f(x)$ has discontinuity at $x = 0, 1/2, 1/2^2, 1/2^3 \ldots$

Let, $S = \{0, 1/2, 1/2^2, 1/2^3, \ldots\}$

Clearly, S has only one limit point, namely 0.

Since, the number of limit points of S is finite,

$\therefore f(x) \in R[0,1]$.

(VIII) A result :-

Let, (M, m), (M', m') and (M'', m'') be the bounds of $f(x)$, $g(x)$ and $f(x)+g(x)$ respectively on $[a,b]$.

Then,

$$M'' - m'' \le (M - m) + (M' - m')$$

Properties of integrable function :-

Property - I :- Let, $f(x), g(x) \in R[a,b]$. Then $f(x)+g(x) \in R[a,b]$.

$\Rightarrow$ Since, $f(x), g(x) \in R[a,b]$, ~~$\exists$ partitions P_1 and P_2 of $[a,b]$~~ corresponding to $\epsilon > 0$, $\exists$ partitions P_1 and P_2 of $[a,b]$ such that

$$U(P_1,f) - L(P_1,f) < \frac{\epsilon}{2} \quad \text{---(i)}$$

$$U(P_2,g) - L(P_2,g) < \frac{\epsilon}{2} \quad \text{---(ii)}$$

Let, $P = P_1 \cup P_2$, i.e. P is common refinement of P_1 and P_2.

Let, $P = \{a = x_0 < x_1 < x_2 < \dots < x_n = b\}$

Let, (M_r, m_r), (M_r', m_r') and (M_r'', m_r'') be the bounds respectively on rth subinterval of $f(x)$, $g(x)$ and $\{f(x)+g(x)\}$ $\Delta_r = [x_{r-1}, x_r]$ of length $\delta_r = x_r - x_{r-1}$.

$\therefore$ we have,

$$(M_r'' - m_r'') \le (M_r - m_r) + (M_r' - m_r')$$

multiplying by δ_r and taking ~~limit~~ sum $r=1$ to n we have

$$\sum_{r=1}^{n} (M_r'' - m_r'')\delta_r \le \sum_{r=1}^{n}(M_r - m_r)\delta_r + \sum_{r=1}^{n}(M_r' - m_r')\delta_r$$

$$\Rightarrow U(P, f+g) - L(P, f+g) \le \{U(P_1,f) - L(P_1,f)\} + \{U(P_2,g) - L(P_2,g)\}$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$\therefore \{f(x)+g(x)\} \in R[a,b]$.

$$\boxed{\begin{aligned} U(P, f+g) - L(P, f+g) &\le \{U(P,f) - L(P,f)\} + \{U(P,g) - L(P,g)\} \\ &< \{U(P_1,f) - L(P_1,f)\} + \{U(P_2,g) - L(P_2,g)\} \\ &< \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon \end{aligned}}$$

[Note :- If P be the refinement of P_1 of the interval $[a,b]$, then $U(P,f) - L(P,f) \le U(P_1,f) - L(P_1,f)$,

i.e. the oscillatory sum becomes smaller and smaller when the partition is more finer.]

Oscillation :- Let, $f(x)$ be bounded on $[a,b]$. Let, M and m be the bounds of $f(x)$ on $[a,b]$. Then $(M-m)$ is called the oscillation of $f(x)$ on $[a,b]$.

Lemma :- Let, $f(x)$ be bounded on $[a,b]$ and M and m be the bounds of $f(x)$ on $[a,b]$. Then

$$M - m = \sup^m \{|f(x_1) - f(x_2)| : \forall x_1, x_2 \in [a,b]\}$$

Theorem :- Let, $f(x) \in R[a,b]$. Then $|f(x)| \in R[a,b]$.

Proof :- Since, $f(x) \in R[a,b]$, $f(x)$ is bounded on $[a,b]$.

$\exists$ a positive real number B such that

$$|f(x)| \le B;\ \forall x \in [a,b]$$

$\therefore$ we have, $\big||f(x)|\big| = |f(x)| \le B;\ \forall x \in [a,b]$.

$\therefore |f(x)|$ is bounded on $[a,b]$.

Since, $f(x) \in R[a,b]$, Corresponding to $\epsilon > 0$, $\exists$ a partition P of $[a,b]$ such that $U(P,f) - L(P,f) < \epsilon$ ——— (i)

Let, P be the partition, $P = \{a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b\}$

Let,

$$M_r = \sup_{x \in \Delta_r}^{m} f(x), \quad m_r = \inf_{x \in \Delta_r}^{m} f(x),$$

$$M_r' = \sup_{x \in \Delta_r}^{m} |f(x)|, \quad m_r' = \inf_{x \in \Delta_r}^{m} |f(x)|$$

where $\Delta_r = [x_{r-1}, x_r]$, $\delta_r = x_r - x_{r-1}$, $r = 1, 2, \ldots, n$

Then we have, $M_r' - m_r' \le M_r - m_r$ $\big[\,||a| - |b|| \le |a-b|\,\big]$

$$\left||f(x_1')| - |f(x_2')|\right| \le |f(x_1') - f(x_2')| \quad \forall x_1', x_2' \in \Delta_r$$

$$\sup^{m}\left\{\left||f(x_1')| - |f(x_2')|\right|\right\} \le \sup^{m}\left\{|f(x_1') - f(x_2')|\right\}$$

$$\Rightarrow M_r' - m_r' \le M_r - m_r$$

$$\therefore \sum_{r=1}^{n} (M_r' - m_r')\delta_r \le \sum_{r=1}^{n} (M_r - m_r)\delta_r$$

$$\Rightarrow U(P, |f|) - L(P, |f|) \le U(P,f) - L(P,f) \quad \underset{[\text{by (i)}]}{< \epsilon}$$

$$\Rightarrow U(P, |f|) - L(P, |f|) < \epsilon.$$

$\therefore |f(x)| \in R[a,b]$.

[<u>Note</u>:— The converse of the above theorem may not be true.

For example, consider the function $f(x) = \begin{cases} -1, & \text{when } x \text{ is rational} \\ 1, & \text{when } x \text{ is irrational} \end{cases}$ $\forall x \in [a,b]$.

we have, $|f(x)| = 1$, $\forall x \in [a,b]$.

Since, constant function is integrable, $|f(x)| \in R[a,b]$.

But ~~for~~ we have, $f(x) \notin R[a,b]$.]

(III) <u>Theorem:-</u> Let, $f(x) \in R[a,b]$. Then $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$.

<u>Proof:-</u> Since, $f(x) \in R[a,b]$.

$\therefore |f(x)| \in R[a,b]$.

$\therefore \exists$ partition $P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$ of $[a,b]$

such that

$$\lim_{\|P\| \to 0} \sum_{r=1}^{n} f(\xi_r)\,\delta_r = \int_a^b f(x)\,dx$$

and $$\lim_{\|P\| \to 0} \sum_{r=1}^{n} |f(\xi_r)|\,\delta_r = \int_a^b |f(x)|\,dx.$$

$\therefore$ we have,

$$\left|\sum_{r=1}^{n} f(\xi_r)\,\delta_r\right| \le \sum_{r=1}^{n} |f(\xi_r)\,\delta_r|$$

$$\Rightarrow \left|\sum_{r=1}^{n} f(\xi_r)\,\delta_r\right| \le \sum_{r=1}^{n} |f(\xi_r)|\,\delta_r.$$

$$\Rightarrow \left|\lim_{\|P\| \to 0} \sum_{r=1}^{n} f(\xi_r)\,\delta_r\right| \le \lim_{\|P\| \to 0} \sum_{r=1}^{n} |f(\xi_r)|\,\delta_r$$

$$\Rightarrow \left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$

Theorem:– Let, $f(x) \in R[a,c]$ and $f(x) \in R[c,b]$. Then $f(x) \in R[a,b]$, where $a<c<b$.

Proof:– Since, $f(x) \in R[a,c]$ and $f(x) \in R[c,b]$, corresponding to $\epsilon > 0$, $\exists$ partitions P_1 of $[a,c]$ and P_2 of $[c,b]$ such that

$$U(P_1,f) - L(P_1,f) < \epsilon/2 \quad \text{—— (i)}$$

$$U(P_2,f) - L(P_2,f) < \epsilon/2 \quad \text{—— (ii)}$$

Let, $P = P_1 \cup P_2$,

Then P is a partition of $[a,b]$.

$\therefore$ we have,

$$U(P,f) - L(P,f) = \{U(P_1,f) - L(P_1,f)\} + \{U(P_2,f) - L(P_2,f)\}$$

$$< \epsilon/2 + \epsilon/2 \quad [\text{by (i) and (ii)}]$$

$$= \epsilon$$

$\therefore U(P,f) - L(P,f) < \epsilon$

$\therefore f(x) \in R[a,b]$.

$\Rightarrow$ Show that if $f(x) \in R[a,c]$ and $f(x) \in R[c,b]$. Then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

$\Rightarrow$ Let, P_1 and P_2 are the partitions of $[a,c]$ and $[c,b]$ respectively.
and $P = P_1 \cup P_2$

Then, $U(P,f) = U(P_1,f) + U(P_2,f)$

$$\Rightarrow \lim_{\|P\|\to 0} U(P,f) = \lim_{\|P_1\|\to 0} U(P_1,f) + \lim_{\|P_2\|\to 0} U(P_2,f) = \lim_{\|P_1\|\to 0} U(P_1,f) + \lim_{\|P_2\|\to 0} U(P_2,f)$$

$$\Rightarrow \overline{\int_a^b} f(x)\,dx = \overline{\int_a^c} f(x)\,dx + \overline{\int_c^b} f(x)\,dx$$

$$\Rightarrow \int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \quad [\because f(x) \in R[a,c] \text{ and } f(x) \in R[c,b] \text{ imply } f(x) \in R[a,b]].$$

2) If $f(x) \in R[a,b]$ and $g(x) \in R[a,b]$. Then show that $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)dx + \int_a^b g(x)\,dx$.

3) If $f(x) \in R[a,b]$, then $\lambda f(x) \in R[a,b]$ and $\int_a^b \lambda f(x)dx = \lambda \int_a^b f(x)dx$, λ being constant.

4) If $f(x) \in R[a,b]$. Then $f(x) \in R[c,d]$, where $a \le c < d \le b$.

5) Let, $f(x) = \frac{1}{2^n}$, $\frac{1}{2^{n+1}} < x \le \frac{1}{2^n}$, $n = 0,1,2,\ldots$

$= 0$, $x = 0$

Then show that $f(x)$ is integrable on $[0,1]$ and evaluate $\int_0^1 f(x)dx$.

$\Rightarrow$ We have,

$$\int_{1/2^n}^{1} f(x)dx = \int_{1/2^n}^{1/2^{n-1}} f(x)dx + \int_{1/2^{n-1}}^{1/2^{n-2}} f(x)dx + \ldots + \int_{1/2^3}^{1/2^2} f(x)dx + \int_{1/2^2}^{1/2} f(x)dx + \int_{1/2}^{1} f(x)dx$$

$$= \int_{1/2}^{1} f(x)\,dx + \int_{1/2^2}^{1/2} f(x)dx + \int_{1/2^3}^{1/2^2} f(x)dx + \ldots + \int_{1/2^{n-1}}^{1/2^{n-2}} f(x)dx + \int_{1/2^n}^{1/2^{n-1}} f(x)\,dx$$

$$= \int_{1/2}^{1} 1\,dx + \int_{1/2^2}^{1/2} \frac{1}{2}dx + \int_{1/2^3}^{1/2^2} \frac{1}{2^2}dx + \ldots + \int_{1/2^n}^{1/2^{n-1}} \frac{1}{2^{n-1}}dx$$

$$= \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2^2}\right) + \frac{1}{2^2}\left(\frac{1}{2^2} - \frac{1}{2^3}\right) + \ldots + \frac{1}{2^{n-1}}\left(\frac{1}{2^{n-1}} - \frac{1}{2^n}\right)$$

$= \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \cdots + \frac{1}{2^{n-1}}$

$\therefore \cancel{\int_0^1 f(x)\,dx = \frac{1}{2} + \frac{1}{}}$

Taking limit $n \to \infty$ on both sides we have,

$$\int_0^1 f(x)\,dx = \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \cdots$$

$$= \frac{1}{2}\left[1 + \frac{1}{2^2} + \left(\frac{1}{2^2}\right)^2 + \cdots\right]$$

$$= \frac{1}{2} \cdot \frac{1}{1 - 1/2^2} = \frac{2}{3}$$

Some inequalities and results :-

1) $m(b-a) \leq \underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx \leq M(b-a)$

2) $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$, if $f(x) \in R[a,b]$.

3) If $f(x)$ be bounded and integrable on $[a,b]$. ~~and~~ If $f(x)$ be bounded and integrable on $[a,b]$, then $\exists$ a real number μ between the bounds of $f(x)$ such that $\int_a^b f(x)\,dx = \mu(b-a)$.

$\Rightarrow$ we have, $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$

$$\Rightarrow m \leq \frac{\int_a^b f(x)\,dx}{b-a} \leq M$$

$$\Rightarrow m \leq \mu \leq M, \text{ where } \mu = \frac{\int_a^b f(x)\,dx}{b-a}$$

$$\therefore \int_a^b f(x)\,dx = \mu(b-a).$$

4) If $f(x)$ be continuous on $[a,b]$, then $\exists$ a point $\xi \in [a,b]$ such that $\int_a^b f(x)dx = f(\xi)(b-a)$.

$\Rightarrow$ Since, $f(x)$ is continuous on $[a,b]$,

$f(x) \in R[a,b]$.

Then we have,

$$\int_a^b f(x)dx = \mu(b-a), \text{ where } m \le \mu \le M.$$

Since, $f(x)$ is continuous on $[a,b]$.

$\therefore \exists$ a point $\xi \in [a,b]$ such that $f(\xi) = \mu$.

$$\therefore \int_a^b f(x)dx = f(\xi)(b-a).$$

5) Let, $f(x) \in R[a,b]$ and $f(x) \geq 0$, $\forall x \in [a,b]$. Then $\int_a^b f(x)dx \geq 0$.

$\Rightarrow$ Since, $f(x) \geq 0$, $\forall x \in [a,b]$, $m \geq 0$.

$\therefore$ We have,

$$\int_a^b f(x)dx \geq m(b-a) \geq 0$$

$$\therefore \int_a^b f(x)dx \geq 0.$$

6) Let, $f(x)$ and $g(x)$ be integrable on $[a,b]$ and $f(x) \geq g(x)$, $\forall x \in [a,b]$. Then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.

7) Let, $f(x)$ be integrable on $[a,b]$ and $f(x)$ is never negative on $[a,b]$. Let, $f(x)$ be continuous at a point $c\in[a,b]$ and $f(c)>0$, then $\int_a^b f(x)dx > 0$.

8) Let, $f(x)$, $g(x)$ be integrable on $[a,b]$ and $f(x)\geq g(x)$, $\forall x\in[a,b]$. Let, $f(x)$ and $g(x)$ be both continuous at a point $c\in[a,b]$ and $f(c)>g(c)$. Then $\int_a^b f(x)dx > \int_a^b g(x)dx$

1) Show that $\frac{\sqrt{3}}{8} \leq \int_{\pi/4}^{\pi/3} \frac{\sin x}{x} dx \leq \frac{\sqrt{2}}{6}$.

$\Rightarrow$ Let, $f(x)=\frac{\sin x}{x}$, $\forall x\in\left[\frac{\pi}{4},\frac{\pi}{3}\right]$.

$\therefore f'(x)=\frac{x\cos x-\sin x}{x^2} = \frac{\cos x\,(x-\tan x)}{x^2} < 0$ in $\left[\frac{\pi}{4},\frac{\pi}{3}\right]$

$\therefore f(x)$ is decreasing in $\left[\frac{\pi}{4},\frac{\pi}{3}\right]$.

$\therefore m=f\left(\frac{\pi}{3}\right)=\frac{\sqrt{3}/2}{\pi/3}=\frac{3\sqrt{3}}{2\pi}$

$\therefore M=f\left(\frac{\pi}{4}\right)=\frac{1/\sqrt{2}}{\pi/4}=\frac{4}{\pi\sqrt{2}}$

$\therefore$ using the result $m(b-a)\leq\int_a^b f(x)dx\leq M(b-a)$ we have,

$$\frac{3\sqrt{3}}{2\pi}\left(\frac{\pi}{3}-\frac{\pi}{4}\right)\leq\int_{\pi/4}^{\pi/3}\frac{\sin x}{x}dx\leq\frac{4}{\pi\sqrt{2}}\left(\frac{\pi}{3}-\frac{\pi}{4}\right)$$

$$\Rightarrow \frac{\cancel{3}\sqrt{3}}{2}\times\frac{1}{\cancel{12}4}\leq\int_{\pi/4}^{\pi/3}\frac{\sin x}{x}dx\leq\frac{\cancel{4}}{\sqrt{2}}\times\frac{1}{\cancel{12}3}$$

$$\Rightarrow \frac{\sqrt{3}}{8}\leq\int_{\pi/4}^{\pi/3}\frac{\sin x}{x}dx\leq\frac{\sqrt{2}}{6}$$

2) Show that $\frac{1}{2} < \int_0^1 \frac{dx}{\sqrt{4-x^2+x^3}} < \frac{\pi}{6}$

$\Rightarrow$ We have,

$4-x^2+x^3 = 4-(x^2-x^3) < 4$ in $0<x<1$

$4-x^2+x^3 = (4-x^2)+x^3 > 4-x^2$

$\therefore 4 > 4-x^2+x^3 > 4-x^2$

$\Rightarrow 2 > \sqrt{4-x^2+x^3} > \sqrt{4-x^2}$

$\Rightarrow \frac{1}{2} < \frac{1}{\sqrt{4-x^2+x^3}} < \frac{1}{\sqrt{4-x^2}}$ in $0<x<1$

$\Rightarrow \frac{1}{2} \le \frac{1}{\sqrt{4-x^2+x^3}} \le \frac{1}{\sqrt{4-x^2}}$ $[0,1]$

$\therefore$ We have,

$$\int_0^1 \frac{1}{2}\,dx < \int_0^1 \frac{dx}{\sqrt{4-x^2+x^3}} < \int_0^1 \frac{dx}{\sqrt{4-x^2}}$$

$$\Rightarrow \frac{1}{2} < \int_0^1 \frac{dx}{\sqrt{4-x^2+x^3}} < \sin^{-1}\frac{x}{2}\Big]_0^1$$

$$\Rightarrow \frac{1}{2} < \int_0^1 \frac{dx}{\sqrt{4-x^2+x^3}} < \frac{\pi}{6}$$

(III) Fundamental theorem of integral calculus :—

Statement :– Let,

i) $\int_a^b f(x)dx$ exists

and ii) $\exists$ a function $\phi(x)$ on $[a,b]$ such that

$\phi'(x) = f(x), \ \forall x \in [a,b]$.

Then, $\int_a^b f(x)dx = \phi(b) - \phi(a)$.

Proof :– Let, $P = \{a = x_0 < x_1 < x_2 < \dots < x_{r-1} < x_r < \dots < x_{n-1} < x_n = b\}$ be a partition of $[a,b]$. ~~with $\|P\|$~~

We have, $\phi(x_r) - \phi(x_{r-1}) = \phi'(\xi_r)\,\delta_r \ ; \ \xi_r \in]x_{r-1}, x_r[$.

$\therefore \sum_{r=1}^{n} \{\phi(x_r) - \phi(x_{r-1})\} = \sum_{r=1}^{n} \phi'(\xi_r)\,\delta_r$ [$\delta_r = x_r - x_{r-1}$ and applying MVT]

$\Rightarrow \phi(b) - \phi(a) = \sum_{r=1}^{n} \phi'(\xi_r)\,\delta_r$

$\Rightarrow \lim_{\|P\|\to 0} (\phi(b) - \phi(a)) = \lim_{\|P\|\to 0} \sum_{r=1}^{n} \phi'(\xi_r)\,\delta_r$

$\Rightarrow \phi(b) - \phi(a) = \int_a^b f(x)\,dx$ [$\because \int_a^b f(x)\,dx$ exists]

[Note :– The function $\phi(x)$ is called primitive of $f(x)$.]

[A function $\phi(x)$ is said to be the primitive of $f(x)$ on $[a,b]$ if the derivative of $\phi(x)$ is equal to $f(x)$, for each $x \in [a,b]$.
i.e. $\phi'(x) = f(x), \forall x \in [a,b]$. $\phi(x) = \int_a^x f(t)\,dt$]

2) Let, P be a partition of $[a,b]$.
Then we have,

~~$U(P,f) \le U(P$~~

$U(P, f+g) \le U(P,f) + U(P,g)$ —— (i)

and $L(P, f+g) \geq L(P,f) + L(P,g)$ —— (ii)

from (i),

$$\lim_{||P|| \to 0} U(P, f+g) \le \lim_{||P|| \to 0} U(P,f) + \lim_{||P|| \to 0} U(P,g)$$

$$\Rightarrow \overline{\int_a^b}(f(x)+g(x))\,dx \le \overline{\int_a^b} f(x)\,dx + \overline{\int_a^b} g(x)\,dx$$

$$\Rightarrow \int_a^b (f(x)+g(x))\,dx \le \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \text{ —— (iii)} \quad \left[\because f(x), g(x) \in R[a,b] \Rightarrow f(x)+g(x) \in R[a,b]\right]$$

from (ii),

$$\lim_{||P|| \to 0} L(P, f+g) \geq \lim_{||P|| \to 0} L(P,f) + \lim_{||P|| \to 0} L(P,g)$$

$$\Rightarrow \underline{\int_a^b}(f(x)+g(x))\,dx \geq \underline{\int_a^b} f(x)\,dx + \underline{\int_a^b} g(x)\,dx$$

$$\Rightarrow \int_a^b (f(x)+g(x))\,dx \geq \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \text{ —— (iv)}$$

from (iii) and (iv), $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

3) Since, $f(x) \in R[a,b]$, $f(x)$ is bounded on $[a,b]$.

$\therefore \exists B > 0$ such that, $|f(x)| \le B, \forall x \in [a,b]$.

Now, $|\lambda f(x)| = \lambda |f(x)| \le \lambda B$, $\forall x \in [a,b]$

$\therefore \lambda f(x)$ is bounded on $[a,b]$.

Let, ~~$M_r = \sup$~~ P be the partition of $[a,b]$,

$P = \{a = x_0 < x_1 < x_2 < \dots < x_n = b\}$

Let, $M_r = \sup\limits_{x \in \Delta_r}^{m} f(x)$, $m_r = \inf\limits_{x \in \Delta_r}^{m} f(x)$, where $\Delta_r = [x_{r-1}, x_r]$,

$M_r = \sup\limits_{x \in \Delta_r}^{m} \lambda f(x)$, $m_r = \inf\limits_{x \in \Delta_r}^{m} \lambda f(x)$, $\delta_r = x_r - x_{r-1}$, $r = 1, 2, \dots, n$

$= \lambda M_r$ $\qquad = \lambda m_r$

Since, $f(x) \in R[a,b]$.

$\therefore$ for the Partition P of $[a,b]$ we have,

$$U(P,f) - L(P,f) < \epsilon/\lambda \quad \text{——(i)}$$

Now, $U(P,\lambda f) - L(P,\lambda f) = \sum_{r=1}^{n} (M_r' - m_r')\delta_r = \lambda \sum_{r=1}^{n} (M_r - m_r)\delta_r$

$$= \lambda \{U(P,f) - L(P,f)\}$$

$$< \lambda \cdot \epsilon/\lambda \quad [\text{by (i)}] = \epsilon$$

$\therefore U(P,\lambda f) - L(P,\lambda f) < \epsilon$

$\therefore \lambda f(x) \in R[a,b]$.

(iii) Case-I :- Let, $\lambda = 0$

Then $\lambda f(x) = 0$

$\therefore \int_a^b \lambda f(x)\,dx = \int_a^b 0 \cdot dx = 0\cdot(b-a) = 0 = 0\int_a^b f(x)dx = \lambda \int_a^b f(x)dx$.

Case-II :- Let, $\lambda > 0$.

Then, ~~...~~ for any partition P of $[a,b]$, $U(P,\lambda f) = \lambda U(P,f)$

$\therefore \overline{\int_a^b} \lambda f(x)\,dx = \inf^m \{U(P,\lambda f) : P \in P[a,b]\}$

$= \inf^m \{\lambda U(P,f) : P \in P[a,b]\}$

$= \lambda \inf^m \{U(P,f) : P \in P[a,b]\}$

$= \lambda \overline{\int_a^b} f(x)dx$

$\Rightarrow \int_a^b \lambda f(x)dx = \lambda \int_a^b f(x)dx$ $[\because f(x) \in R[a,b] \Rightarrow \lambda f(x) \in R[a,b]]$

Case-III :- Let, $\lambda < 0$

Then let, $\lambda = -\mu$, where $\mu > 0$ $\left[\because \int_a^b -f(x)dx = -\int_a^b f(x)dx\right]$

$\therefore \int_a^b \lambda f(x)dx = \int_a^b (-\mu) f(x)dx = -\int_a^b \mu f(x)dx$

$= -\mu \int_a^b f(x)dx$ $[\because \mu > 0$, by Case II$]$

$= \lambda \int_a^b f(x)dx$

$\therefore$ Combining Cases I, II and III,

$$\int_a^b \lambda f(x)dx = \lambda \int_a^b f(x)dx.$$

4) Since, $f(x) \in R[a,b]$.

$\therefore$ Corresponding to $\epsilon > 0$, $\exists\ \delta > 0$ such that $U(P,f) - L(P,f) < \epsilon$ for all partitions P of $[a,b]$ with $\|P\| < \delta$ ——(i)

Let, P' be the partition of $[c,d]$.

Clearly, $U(P,f) - L(P,f)$ contains all points terms of $U(P',f) - L(P',f)$ and some extra non-negative terms.

$\therefore$ we have,

$$U(P',f) - L(P',f) \leq U(P,f) - L(P,f) < \epsilon \quad [\text{by (i)}]$$

$$\therefore U(P',f) - L(P',f) < \epsilon$$

$\therefore f(x) \in R[c,d]$. (Proved).

<u>Inequalities:-</u>

6) Let, $\varphi(x) = f(x) - g(x)$, $\forall x \in [a,b]$.

Since, $f(x), g(x) \in R[a,b]$.

$\therefore \varphi(x) \in R[a,b]$.

Since, $f(x) \geq g(x)$, $\forall x \in [a,b]$.

$\therefore \varphi(x) \geq 0$, $\forall x \in [a,b]$.

Hence, $\int_a^b \varphi(x)\,dx \geq 0$

$$\Rightarrow \int_a^b (f(x) - g(x))\,dx \geq 0$$

$$\Rightarrow \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \geq 0$$

$$\Rightarrow \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx \quad \text{(Proved)}$$

4) Since, f is bounded on $[a,b]$, it is bounded on $[c,d]$.

Since, f is integrable on $[a,b]$.

$\therefore \exists$ a partition P of $[a,b]$ such that $U(P,f) - L(P,f) < \epsilon$ ——(i)

Let, $P^* = P \cup \{c,d\}$

$\therefore P^*$ be a refinement of P.

$\therefore$ we have, $U(P^*,f) - L(P^*,f) \leq U(P,f) - L(P,f) < \epsilon$ [by (i)]

$$\therefore U(P^*,f) - L(P^*,f) < \epsilon \text{ ——(ii)}$$

Let, $P' = P^* \cap [c,d]$.

$\therefore P'$ be a partition of $[c,d]$.

$\therefore$ Any subinterval of P' is also a subinterval of ~~[c,d]~~ P^*

$\therefore U(P^*,f) - L(P^*,f)$ contains all the terms of $U(P',f) - L(P',f)$ and also some extra point terms.

$\therefore$ We have, $U(P',f) - L(P',f) \le U(P^*,f) - L(P^*,f) < \epsilon$ [by (ii)]

$\Rightarrow U(P',f) - L(P',f) < \epsilon$

$\therefore f(x) \in R[c,d]$. (Proved)

Inequality:-

7) Case - I :- Let, $a<c<b$ and $\epsilon = \frac{1}{2} f(c) > 0$ as $f(c) > 0$.

$\because f$ is continuous at c.

$\therefore$ for $\epsilon > 0$, $\exists\, \delta > 0$ s.t. $|f(x) - f(c)| < \epsilon$ $\forall x \in [c-\delta, c+\delta]$

$\Rightarrow f(x) > \frac{1}{2} f(c)$, $\forall x \in [c-\delta, c+\delta]$

$\because f$ is non-negative and integrable on $[a, c-\delta]$ and $[c+\delta, b]$.

$\therefore \int_a^{c-\delta} f(x)dx \geq 0$ and $\int_{c+\delta}^b f(x)dx \geq 0$

and f is integrable on $[c-\delta, c+\delta]$

Again since, $f(x) > \frac{1}{2} f(c)$, $\forall x \in [c-\delta, c+\delta]$

$\therefore \int_{c-\delta}^{c+\delta} f(x)dx \geq \int_{c-\delta}^{c+\delta} \frac{1}{2} f(c)dx = \frac{1}{2} f(c) 2\delta > 0$

Now, $\int_a^b f(x)dx = \int_a^{c-\delta} f(x)dx + \int_{c-\delta}^{c+\delta} f(x)dx + \int_{c+\delta}^b f(x)dx > 0$.

Case - II :- Let, $a = c$ and $\epsilon = \frac{1}{2} f(a) > 0$.

$\because f$ is continuous at a, for $\epsilon > 0$, $\exists\, \delta > 0$ s.t.

$|f(x) - f(a)| < \epsilon$, $\forall x \in [a, a+\delta]$

$\Rightarrow f(x) > \frac{1}{2} f(a)$, $\forall x \in [a, a+\delta]$

and since, $f(x)$ is integrable on $[a, a+\delta]$

$\therefore \int_a^{a+\delta} f(x)dx \geq \int_a^{a+\delta} \frac{1}{2} f(a)dx = \frac{1}{2} f(a)\cdot\delta > 0$

Again since, $f(x)$ is non-negative on $[a+\delta, b]$ and integrable on $[a+\delta, b]$

$\therefore \int_{a+\delta}^b f(x)dx \geq 0$

Now, $\int_a^b f(x)dx = \int_a^{a+\delta} f(x)dx + \int_{a+\delta}^b f(x)dx > 0$

$\therefore \int_a^b f(x)dx > 0$

Case - III :- ~~Similar~~ Let, $b = c$.

Then similar is the case as case - II.

(III) First M.V.T. of Integral Calculus :-

Statement :- Let, $f(x)$ and $\varphi(x)$ be two bounded and integrable functions on $[a, b]$ and $\varphi(x)$ maintains the same sign on $[a, b]$, then $\exists$ a real number μ such that $\int_a^b f(x)\varphi(x)dx = \mu \int_a^b \varphi(x)dx$, where $m \le \mu \le M$, m and M are the bounds of $f(x)$ on $[a, b]$.

Proof :- Let, $\varphi(x)$ be non-negative on $[a, b]$.

We have,

$$m \le f(x) \le M$$

$$\Rightarrow m\varphi(x) \le f(x)\varphi(x) \le M\varphi(x)$$

$$\Rightarrow \int_a^b m\varphi(x)\,dx \le \int_a^b f(x)\varphi(x)\,dx \le \int_a^b M\varphi(x)\,dx$$

$$\Rightarrow m\int_a^b \varphi(x)dx \le \int_a^b f(x)\varphi(x)\,dx \le M\int_a^b \varphi(x)dx$$

$$\Rightarrow m \le \frac{\int_a^b f(x)\varphi(x)dx}{\int_a^b \varphi(x)dx} \le M$$

$$\therefore m \le \mu \le M$$

where $\mu = \dfrac{\int_a^b f(x)\varphi(x)dx}{\int_a^b \varphi(x)dx}$

$$\therefore \int_a^b f(x)\,\varphi(x)\cdot dx = \mu\int_a^b \varphi(x)dx.$$

Similar is the case when $\varphi(x)$ is negative.

[Note:- i) If moreover $f(x)$ be continuous on $[a, b]$, then $\exists$ a point $\xi \in [a, b]$ such that $\int_a^b f(x)\varphi(x)\,dx = f(\xi)\int_a^b \varphi(x)\,dx$.

ii) Let, $f(x)$ be bounded and integrable on $[a, b]$. Then $\exists$ a real number μ between the bounds of $f(x)$ such that $\int_a^b f(x)\,dx = \mu(b-a)$.

This is sometimes known as first M.V.T. of integral calculus and the above theorem is it's generalisation.

iii) If $f(x)$ be continuous on $[a, b]$, then $\exists$ a point $\xi \in [a, b]$ such that $\int_a^b f(x)\,dx = f(\xi)(b-a)$.]

1) Show that for $k^2 < 1$, $\frac{\pi}{6} \le \int_0^{1/2} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \le \frac{\pi}{6} \cdot \frac{1}{\sqrt{1-k^2/4}}$.

$\Rightarrow$ Let, $f(x) = \frac{1}{\sqrt{1-k^2x^2}}$ and $\varphi(x) = \frac{1}{\sqrt{1-x^2}}$.

Clearly, $f(x)$ and $\varphi(x)$ satisfies all the conditions of first M.V.T. of integral calculus.

$\therefore \exists$ a point $\xi \in [0, \frac{1}{2}]$ such that

$$\int_0^{1/2} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{1}{\sqrt{1-k^2\xi^2}} \int_0^{1/2} \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{6} \cdot \frac{1}{\sqrt{1-k^2\xi^2}}$$

Putting $\xi = 0$ and $\xi = \frac{1}{2}$ we have,

$$\frac{\pi}{6} \le \int_0^{1/2} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \le \frac{\pi}{6} \frac{1}{\sqrt{1-\frac{k^2}{4}}}$$

(III) **Abel's inequality :-**

Let, i) $a_1, a_2, \ldots, a_n$ are non-increasing positive real numbers.

ii) $v_1, v_2, \ldots, v_n$ be any n real numbers.

and iii) h and H be two real numbers such that $h \leq v_1 + v_2 + \ldots + v_p \leq H$, $1 \leq p \leq n$.

Then

$$a_1 h \leq \sum_{r=1}^{n} a_r v_r \leq a_1 H$$

(III) **2nd M.V.T. of integral calculus (Bonnet's form) :-**

Statement :- Let, $f(x)$ be bounded, monotonic non-increasing and never negative on $[a, b]$; and $\varphi(x)$ be bounded and integrable on $[a, b]$. Then $\exists$ a point $\xi \in [a, b]$ such that

$$\int_a^b f(x)\varphi(x)\,dx = f(a)\int_a^{\xi} \varphi(x)\,dx.$$

Proof :- Let, $P = \{a = x_0 < x_1 < x_2 < \ldots < x_{r-1} < x_r < \ldots < x_{n-1} < x_n = b\}$ be a partition of $[a, b]$.

Let, $m_r = \inf\limits_{x \in \Delta_r}{}^{m} \varphi(x)$ and $M_r = \sup\limits_{x \in \Delta_r}{}^{m} \varphi(x)$

where, $\Delta_r = [x_{r-1}, x_r]$ and $\delta_r = x_r - x_{r-1}$, $r = 1, 2, \ldots, n$

Let, $a = \xi_1$ and $\xi_r \in \Delta_r$ be arbitrary $(r \neq 1)$

$\therefore$ we have,

$$m_r \delta_r \le \int_{x_{r-1}}^{x_r} \varphi(x)\,dx \le M_r \delta_r$$

and $$m_r \delta_r \le \varphi(\xi_r)\delta_r \le M_r \delta_r$$

Putting $r = 1, 2, \ldots, p$, where $p \le n$ and adding we have,

$$\sum_{r=1}^{p} m_r \delta_r \le \int_a^{x_p} \varphi(x)\,dx \le \sum_{r=1}^{p} M_r \delta_r$$

and $$\sum_{r=1}^{p} m_r \delta_r \le \sum_{r=1}^{p} \varphi(\xi_r)\delta_r \le \sum_{r=1}^{p} M_r \delta_r$$

$$\therefore \left| \sum_{r=1}^{p} \varphi(\xi_r)\delta_r - \int_a^{x_p} \varphi(x)\,dx \right| \le \sum_{r=1}^{p} (M_r - m_r)\delta_r$$

$$\le \sum_{r=1}^{n} (M_r - m_r)\delta_r$$

$$\therefore \int_a^{x_p} \varphi(x)\,dx - \sum_{r=1}^{n} (M_r - m_r)\delta_r \le \sum_{r=1}^{p} \varphi(\xi_r)\delta_r \le \int_a^{x_p} \varphi(x)\,dx + \sum_{r=1}^{n} (M_r - m_r)\delta_r \quad \text{(i)}$$

Now, $\int_a^{x} \varphi(x)\,dx$ is continuous and hence bounded on $[a, b]$.

Let, m and M be the bounds of $\int_a^{x} \varphi(x)\,dx$ on $[a, b]$.

$$\therefore m - \sum_{r=1}^{n} (M_r - m_r)\delta_r \le \sum_{r=1}^{p} \varphi(\xi_r)\delta_r \le M + \sum_{r=1}^{n} (M_r - m_r)\delta_r \quad \text{(ii)}$$

Now let,

$a_r = f(\xi_r)$, $v_r = \varphi(\xi_r)\delta_r$,

$$h = m - \sum_{r=1}^{n}(M_r - m_r)\delta_r,$$

$$H = M + \sum_{r=1}^{n}(M_r - m_r)\delta_r$$

$\therefore$ By Abel's inequality we have,

$$f(a)\left\{m - \sum_{r=1}^{n}(M_r - m_r)\delta_r\right\} \leq \sum_{r=1}^{n} f(\xi_r)\varphi(\xi_r)\delta_r$$
$$\leq f(a)\left\{M + \sum_{r=1}^{n}(M_r - m_r)\delta_r\right\}$$

Taking $||P|| \to 0$ we have,

$$m f(a) \leq \int_a^b f(x)\varphi(x)\,dx \leq M f(a).$$

$$\therefore \int_a^b f(x)\varphi(x)\,dx = f(a)\cdot \mu, \text{ where } m \leq \mu \leq M.$$

Again since, $\int_a^x \varphi(x)\,dx$ is continuous and

m, M are ~~the bou~~ it's bounds

$\therefore \exists$ a point $\xi \in [a, b]$ such that $\mu = \int_a^{\xi} \varphi(x)\,dx$

$$\therefore \int_a^b f(x)\,\varphi(x)\,dx = f(a)\int_a^{\xi}\varphi(x)\,dx$$

2nd M.V.T. of integral calculus : Weierstrass form :-

Statement :- Let, $f(x)$ be bounded and monotonic ~~non-increasing~~ on $[a,b]$; and $\varphi(x)$ be bounded and integrable on $[a,b]$.

Then $\exists$ a point $\xi \in [a,b]$ such that

$$\int_a^b f(x)\varphi(x)\,dx = f(a)\int_a^{\xi}\varphi(x)\,dx + f(b)\int_{\xi}^{b}\varphi(x)\,dx.$$

Proof :- Let, $f(x)$ be M.I.

$\therefore f(b)-f(x)$ is M.D. and never negative on $[a,b]$.

$\therefore$ By Bonnet's form,

$$\int_a^b \big(f(b)-f(x)\big)\varphi(x)\,dx = \big(f(b)-f(a)\big)\int_a^{\xi}\varphi(x)\,dx$$

$\therefore$ ~~$\int_a^b f(x)\varphi(x)\,dx$~~

$$\therefore \int_a^b f(b)\varphi(x)\,dx - \int_a^b f(x)\varphi(x)\,dx = f(b)\int_a^{\xi}\varphi(x)\,dx - f(a)\int_a^{\xi}\varphi(x)\,dx$$

$$\Rightarrow \int_a^b f(x)\varphi(x)\,dx = f(a)\int_a^{\xi}\varphi(x)\,dx + f(b)\int_{\xi}^{b}\varphi(x)\,dx.$$

**

1) Show that $\left|\int_{x'}^{x''}\frac{\sin x}{x}\,dx\right| \le \frac{2}{x'}$, $0<x'<x''$.

$\Rightarrow$ Let, $f(x)=\frac{1}{x}$ and $\varphi(x)=\sin x$ on $[x', x'']$.

Clearly, $f(x)$ and $\varphi(x)$ satisfies all the conditions of 2nd M.V.T. in Bonnet's form.

$\therefore$ we have, $\int_{x'}^{x''}\frac{\sin x}{x}\,dx = \frac{1}{x'}\int_{x'}^{\xi}\sin x\,dx$, $\xi \in [x', x'']$

$= \cancel{\frac{1}{x'}}$

$$\left|\int_{x'}^{x''}\frac{\sin x}{x}\,dx\right| = \frac{1}{x'}\left|\int_{x'}^{\xi}\sin x\,dx\right|$$

$$= \frac{1}{x'}\left|\cos x' - \cos\xi\right|$$

$$\leq \frac{1}{x'}\left\{|\cos x'| + |\cos\xi|\right\}$$

$$\leq \frac{1}{x'}(1+1) \quad [\because |\cos\theta| \leq 1]$$

$$= \frac{2}{x'}$$

2) Show that $\left|\int_{x'}^{x''}\frac{\sin x}{x}\,dx\right| \leq \frac{4}{x'}$, $0 < x' < x''$.

$\Rightarrow$ Let, $f(x) = \frac{1}{x}$ and $\phi(x) = \sin x$ on $[x', x'']$

clearly, $f(x)$ and $\phi(x)$ satisfies all the conditions of 2nd M.V.T. of integral calculus of Weierstrass form.

$\therefore$ we have,

$$\int_{x'}^{x''}\frac{\sin x}{x}\,dx = \frac{1}{x'}\int_{x'}^{\xi}\sin x\,dx + \frac{1}{x''}\int_{\xi}^{x''}\sin x\,dx$$

$$\Rightarrow \left|\int_{x'}^{x''}\frac{\sin x}{x}\,dx\right| \leq \frac{1}{x'}\left|\int_{x'}^{\xi}\sin x\,dx\right| + \frac{1}{x''}\left|\int_{\xi}^{x''}\sin x\,dx\right|$$

$$= \frac{1}{x'}\left|\cos x' - \cos\xi\right| + \frac{1}{x''}\left|\cos\xi - \cos x''\right|$$

$$\leq \frac{2}{x'} + \frac{2}{x''} \quad \left[\because x' < x'' \Rightarrow \frac{1}{x''} < \frac{1}{x'} \Rightarrow \frac{2}{x''} < \frac{2}{x'}\right]$$

$$\leq \frac{2}{x'} + \frac{2}{x'}$$

$$= \frac{4}{x'}$$

$$\therefore \left|\int_{x'}^{x''}\frac{\sin x}{x}\,dx\right| \leq \frac{4}{x'}$$